Lecture 9

The Schwarzschild solution

Recap

Where were we?

Newtonian Gravity

① $\boxed{\vec{a} = -\vec{\nabla}\Phi}$ grav. pot.

② $\boxed{\nabla^2 \Phi = 4\pi G \rho}$ mass density

Newton's constant

How to generalize these equations?

Minimal-coupling principle = just write SR laws in coord. inv. form!

Example: particles move in straight lines in Minkowski spacetime

$$\frac{d^2 x^\mu}{d\tau^2} = 0 \iff \frac{dx^\nu}{d\tau}\partial_\nu \frac{dx^\mu}{d\tau} = 0 \rightarrow \boxed{\frac{dx^\nu}{d\tau}\nabla_\nu \frac{dx^\mu}{d\tau} = 0}$$

geodesic equation.

Newtonian limit

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$

$v \ll c$, $\partial_t h_{\mu\nu} \simeq 0$

$\boxed{g_{00} = -1 - 2\Phi}$

①

Einstein Equation

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}} \underset{n=4}{\Leftrightarrow} \boxed{R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T^{\alpha}{}_{\alpha} g_{\mu\nu} \right)}$$

$\mu=\nu=0$ $\downarrow$ Newtonian limit

$$\boxed{\nabla^2 \Phi \simeq 4\pi G \rho}$$

Lagrangian formulation of GR

Hilbert action :

$$\boxed{S_H = \int d^n x \sqrt{-g}\, R}$$

$$\delta S_H = \int d^n x \sqrt{-g} \underbrace{\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right)}_{\overset{!}{=}\,0} \delta g^{\mu\nu}$$

Einstein equation in vacuum

In general, the full action is

$$S = \frac{1}{16\pi G} S_H + \underbrace{S_M}_{\text{matter}}$$

$$\delta S = \int d^n x \sqrt{-g}\, \delta g^{\mu\nu} \left[\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) + \underbrace{\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}}_{\equiv -\frac{1}{2} T_{\mu\nu}} \right] = 0$$

We recover E. Eqs. if we define

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}}$$

- symmetric ✓
- conserved ($\nabla^\mu T_{\mu\nu} = 0$) ✓ (to be explained)

Exercise: explore this def. in examples (EM fields, scalar fields, ...)

Plan for today

- Einstein - Hilbert Action

- The Cosmological Constant

- GR as an Effective Field Theory (EFT)

- The Schwarzschild metric

The cosmological constant

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} (R - 2\Lambda) + \underbrace{\hat{\mathcal{L}}_M}_{\text{Matter Lagrangian density}} \right]$$

$$= \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \hat{\mathcal{L}}_M - \underbrace{\rho_{vac}}_{\text{vacuum energy density}} \right]$$

$$\rho_{vac} = \frac{\Lambda}{8\pi G}$$

Exercise: show that

$$ds^2 = - dt^2 + e^{2Ht} \overbrace{(d\vec{x})^2}^{\delta_{ij}\, dx^i dx^j}$$

[de Sitter spacetime]

solves E. eqs. without matter ($\hat{\mathcal{L}}_M = 0$) and

$$\Lambda = H^2 \; \#$$

determine this number

The cosmological constant

Vacuum energy density

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \hat{\mathcal{L}}_M - \rho_{vac} \right] \qquad \rho_{vac} = \frac{\Lambda}{8\pi G}$$

What is the value of ρ_{vac} ?

* Theory : $[\rho_{vac}] = (\text{Energy})^4 \rightarrow \rho_{vac}^{theo} \sim M_p^4 \sim (10^{19}\,\text{GeV})^4$

  $M_{LHC}^4 \sim (10^4\,\text{GeV})^4$

* Experiment : $\rho_{vav}^{exp} \approx (2 \times 10^{-12}\,\text{GeV})^4$

$$\frac{\rho_{vac}^{theo}}{\rho_{vac}^{exp}} \sim 10^{123}$$

Cosmological constant problem

General Relativity as an Effective Field Theory

Most general worldline action :

non-minimal couplings

$$S = -m\int d\tau \left[1 + c_1 R + c_2 R_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \dots \right] , \quad \bullet \equiv \frac{d}{d\tau}$$

$$[c_1] = [c_2] = L^2$$

What should be the values of $c_1, c_2, \dots$?

$$c_1 \sim c_2 \sim L^2 \quad \text{with } L = \text{microscopic length} = \begin{cases} l_p \sim 10^{-35} \, m \\ \lambda_{\text{Compton}} = \dfrac{h}{mc} \\ ?? \end{cases}$$

But

$$R_{\mu\nu} \sim \partial_\mu \partial_\nu \Phi \sim \frac{GM}{r^3} \sim \left\{ \begin{matrix} 10^{-24} \, m^{-2} \text{ for Sun} \\ \\ 10^{-23} \, m^{-2} \text{ for Earth} \end{matrix} \right\} \sim (10^{12} \, m)^{-2}$$

$$\Rightarrow \quad c_1 R \ll 1 \quad \underline{\text{not observable}}$$

General Relativity as an Effective Field Theory

Similarly,

$$S = S_M + \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\left[R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 \nabla_\mu R \nabla^\mu R + \dots\right]$$

$$\alpha_1 \sim \alpha_2 \sim L^2 \quad \text{with } L = \text{microscopic length} \sim \begin{cases} l_P \sim 10^{-35}\,m \\ L_{ov} \lesssim 10^{-20}\,m \\ \quad (\text{LHC reach}) \end{cases}$$

But

$$R_{\mu\nu} \sim \partial_\mu \partial_\nu \Phi \sim \frac{GM}{r^3} \sim \left\{ \begin{matrix} 10^{-24}\,m^{-2} & \text{for Sun} \\ 10^{-23}\,m^{-2} & \text{for Earth} \end{matrix} \right\} \sim \left(10^{12}\,m\right)^{-2}$$

$$\Rightarrow \quad \alpha_1 R \lesssim 10^{-60} \ll 1 \qquad \underline{\text{not observable}}$$

General Relativity is non-linear

$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R \quad + S_M$$

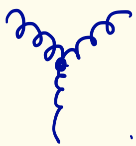
with $\quad R \sim \partial\Gamma + \Gamma\Gamma \quad , \quad \Gamma \sim g^{-1}\partial g$

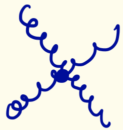
$\sqrt{16\pi G} \sim \frac{1}{M_p}$

Fluctuations around Minkowski spacetime : $\quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$

$$S = \int d^4x \left[h\partial^2 h + \kappa\, h\partial h \partial h + \kappa^2 h^2 (\partial h)^2 + \dots \right]$$

graviton propagator



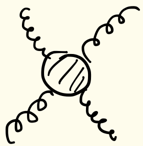


interaction vertices

[Contrast this with EM action]

$\kappa^2 (\text{Energy})^2 \sim \left(\frac{\text{Energy}}{M_p}\right)^2$

Graviton scattering :



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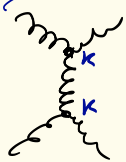
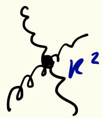


+ perms. +



+ loops

Free discussion

The Schwarzschild solution

Consider first the analogous solution in Newtonian gravity:

mass density

$$\circledast \quad \boxed{\nabla^2 \Phi = 4\pi G \rho}$$

Newton's constant

Vacuum + Spherical symmetry $\Rightarrow \partial_r\left(r^2 \partial_r \Phi\right) = 0 \Rightarrow \Phi(r) = \frac{c_1}{r} + \cancel{c_2}$ 0 by $\Phi(\infty)=0$

$(\rho = 0)$

To determine c_1, integrate $\circledast$ inside a ball of radius R:

$$4\pi G M = \int_{r<R} d^3r \, \vec{\nabla} \cdot \vec{\nabla} \Phi = 4\pi R^2 \, \vec{e}_r \cdot \vec{\nabla} \Phi \Big|_{r=R} = -4\pi c_1$$

$$\Rightarrow \quad \boxed{\Phi(r) = -\frac{GM}{r}} \quad \text{for } r > R_{\text{body}}$$

The Schwarzschild metric is the unique spherically symmetric (and static) vacuum solution of Einstein equations.

Static + Spherical symmetry $\Rightarrow$ $ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + \cancel{e^{2\gamma(r)}}\, r^2 \overbrace{d\Omega^2}^{d\theta^2 + \sin^2\theta\, d\phi^2}$

by choice of radial coord.

$$\begin{cases} \tilde{r} = e^{\gamma(r)} r \\ e^{2\alpha(r)} = e^{2\tilde{\alpha}(\tilde{r})} \\ e^{\beta(r)}dr = e^{\tilde{\beta}(\tilde{r})}d\tilde{r} \end{cases}$$

"drop tildes"

$\boxed{R_{\mu\nu} = 0}$ leads to eqs. for $\alpha(r)$ and $\beta(r)$.

Exercise: show that

$$' \equiv \frac{d}{dr}$$

$$R_{tt} = e^{2(\alpha-\beta)}\left[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha'\right]$$

$$R_{rr} = -\alpha'' - \alpha'^2 + \alpha'\beta' + \frac{2}{r}\beta'$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\beta'-\alpha') - 1\right] + 1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

Notice that:

$$0 = e^{2(\beta-\alpha)} R_{tt} + R_{rr} = \frac{2}{r}(\alpha'+\beta') \Rightarrow \alpha(r) = -\beta(r) + \cancelto{0}{c} \quad \text{using} \quad t \to e^{-c} t$$

then

$$0 = R_{\theta\theta} = -e^{2\alpha}(2r\alpha'+1) + 1 \iff \left(r e^{2\alpha}\right)' = 1 \iff \boxed{e^{2\alpha} = 1 - \frac{R_s}{r}}$$

int. const.

Therefore, the Schwarzschild metric is

$$\boxed{ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{R_s}{r}} + r^2\, d\Omega^2}$$

<u>Exercise</u>: check the remaining E. eqs., $R_{rr}=0$.

The parameter R_s is called the Schwarzschild radius.

Notice that $ds^2 \xrightarrow[r\to\infty]{} -dt^2 + dr^2 + r^2\, d\Omega^2$ (Minkowski)

Moreover,

$$g_{00} = -1 + \frac{R_s}{r} \underset{r \gg R_s}{\approx} -1 - 2\Phi \Rightarrow \Phi = -\frac{R_s}{2r} = -\frac{GM}{r}$$

$$\Rightarrow \boxed{R_s = 2GM}$$

Singularities

$$\boxed{ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{R_s}{r}} + r^2\,d\Omega^2}$$

$g_{tt} \xrightarrow[r\to 0]{} \infty$

$g_{rr} \xrightarrow[r\to R_s]{} \infty$

singular points? Not sure because comp. of tensor change under coord. transf.

$$\left[\ ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \ ,\quad g^{\phi\phi} = \frac{1}{\sin^2\theta} \xrightarrow[\theta\to 0]{} \infty\ \right]$$

If a scalar diverges then the geometry is singular

Exercise: show that

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48\, G^2 M^2}{r^6}$$

$r = 0$ curv. sing

$r = R_s$ maybe not

(it is just a coord. sing.)

Stars and planets are not vacuum solutions.

$T_{\mu\nu} \neq 0$ for $r < R_{body}$ but the Schwarzschild metric is still valid fo $r > R_{body}$ (assuming spherical symmetry)

$$\boxed{ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 \, d\Omega^2} \quad \text{for } r > R_{body}$$

Notice that $$R_{sun} = 10^6 \, G M_{sun} \gg 2 G M_{sun} = R_s$$

Lecture 10

Testing GR in the solar system

Recap

Where were we?

General Relativity

Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G\, T_{\mu\nu}} \xrightarrow[\mu=\nu=0]{}$$

Newtonian Limit

$$ds^2 = -(1+2\Phi)\,dt^2 + (1-2\Phi)\,d\vec{x}^{\,2}$$

$$|\Phi| \ll 1\,, \quad \partial_t \Phi = 0\,, \quad p \ll \rho$$

$$\boxed{\nabla^2 \Phi \simeq 4\pi G \rho}$$

Geodesic equations

$$\boxed{\frac{dx^\nu}{d\tau} \nabla_\nu \frac{dx^\mu}{d\tau} = 0} \longrightarrow$$

$$|\vec{v}| \ll c$$

$$\boxed{\cancel{m_{in}} \frac{d^2\vec{x}}{dt^2} = -\cancel{m_g} \vec{\nabla}\Phi}$$

Energy-momentum tensor

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} - \frac{\Lambda}{8\pi G} g_{\mu\nu}}$$

Cosmological constant

$$\frac{\Lambda}{8\pi G} = \rho_{vac}$$

General Relativity

Actions

Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G\, T_{\mu\nu}} \longleftarrow \boxed{S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_M}$$

Geodesic equations

$$\boxed{\frac{dx^\nu}{d\tau} \nabla_\nu \frac{dx^\mu}{d\tau} = 0} \longleftarrow \boxed{S = -m \int d\tau}$$

Energy-momentum tensor

Cosmological constant

$$\frac{\Lambda}{8\pi G} = \rho_{vac}$$

$$\boxed{T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} - \frac{\Lambda}{8\pi G} g_{\mu\nu}}$$

Spherical symmetry + Vacuum ($T_{\mu\nu} = 0$) => The Schwarzschild metric

$$\boxed{ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 \, d\Omega^2}$$

$$\boxed{R_s = 2GM}$$

Stars and planets are not vacuum solutions.

$T_{\mu\nu} \neq 0$ for $r < R_{body}$ but the Schwarzschild metric is still valid for $r > R_{body}$ (assuming spherical symmetry)

Notice that $R_{sun} = 10^6 \, G M_{sun} \gg 2G M_{sun} = R_s$

Plan for today

- Geodesics in the Schwarzschild metric
- Perihelion advance
- Gravitational redshift
- Schwarzschild black holes

Geodesics of Schwarzschild

$$\boxed{ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{R_s}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

In principle, we could directly solve

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

In practice, it is simpler to use the constants of motion:

Killing vector

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{const.}$$

$$-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \epsilon = \begin{cases} 1 & \text{massive} \quad (\lambda = \text{proper time}) \\ 0 & \text{massless} \quad \dfrac{dx^\mu}{d\lambda} = p^\mu \end{cases}$$

Geodesics of Schwarzschild

$$\boxed{ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{R_s}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

Spherical sym $\longrightarrow$ $\theta = \frac{\pi}{2}$ w.l.o.g.

Killing vectors :

$$K^\mu = (\partial_t)^\mu = (1,0,0,0) \longrightarrow K_\mu = \left(-1+\frac{R_s}{r}, 0,0,0\right)$$

$$R^\mu = (\partial_\phi)^\mu = (0,0,0,1) \longrightarrow R_\mu = (0, 0, 0, r^2 \underbrace{\cancel{\sin^2\theta}}_{\theta=\frac{\pi}{2}})$$

Constants of motion :

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = \left(1-\frac{R_s}{r}\right)\frac{dt}{d\lambda}$$

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda}$$

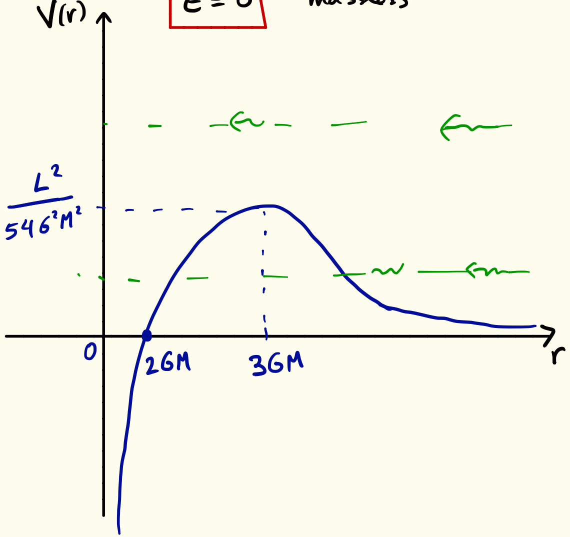
In addition,

$$-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \epsilon \quad \underset{\text{Check!}}{\Longleftrightarrow} \quad \boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}}$$

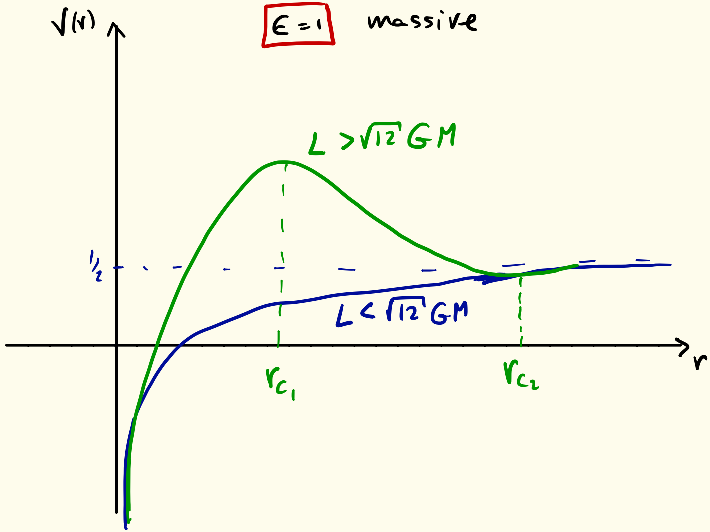
with

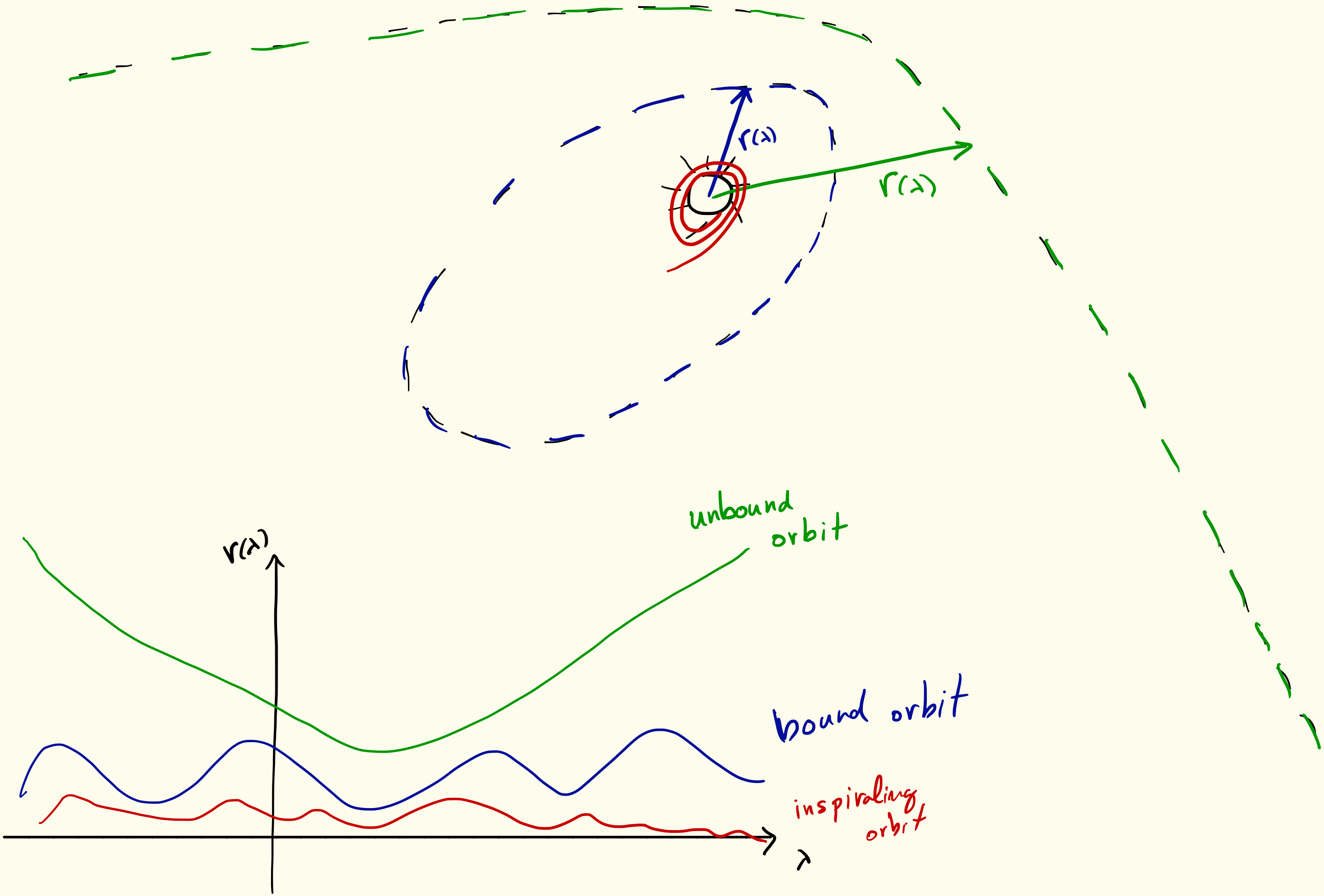
$$V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \underbrace{\frac{GML^2}{r^3}}_{\text{absent in Newtonian gravity}} , \quad \mathcal{E} = \frac{1}{2}E^2$$

$\boxed{\epsilon = 0}$ massless



$\boxed{\epsilon = 1}$ massive





r(λ)
r(λ)
unbound orbit
r(λ)
bound orbit
inspiraling orbit
λ

Circular orbits

$$V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \underbrace{\frac{GML^2}{r^3}}_{\text{absent in Newtonian gravity}}$$

- Massless $\quad \epsilon = 0$

$$V'(r_c) = 0 \iff r_c = 3GM$$

- Massive $\quad \epsilon = 1$

$$V'(r_c) = 0 \iff r_c^2\, GM - L^2 r_c + 3GML^2 = 0$$

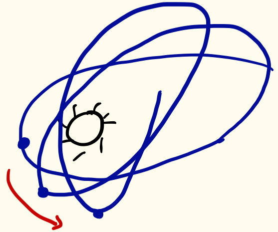
$$\iff r_c = \frac{L^2 \pm L\sqrt{L^2 - 12G^2M^2}}{2GM} \xrightarrow[L\to\infty]{} \begin{cases} \dfrac{L^2}{GM} & \text{stable} \\ 3GM & \text{unstable} \end{cases}$$

$r_{c,\text{stable}} > 6GM$ [minimum value happens for $L^2 = 12G^2M^2$]

If $L < \sqrt{12}\,GM \Rightarrow$ No barrier $\Rightarrow$ particles fall into the BH

Perihelion advance

$\hookrightarrow$ point of closest approach to the Sun



$$\circledast \quad \boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}} \quad , \qquad L = r^2 \frac{d\phi}{d\lambda} \quad , \quad R_s = 2GM$$

$$V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \quad , \quad \mathcal{E} = \frac{1}{2}E^2 \quad , \quad E = \left(1 - \frac{R_s}{r}\right)\frac{dt}{d\lambda}$$

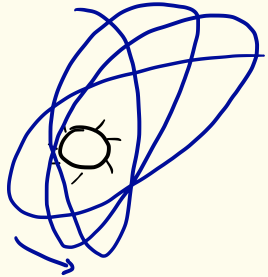
$$\circledast \times \left[\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{r^4}{L^2}\right] \Rightarrow \left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{R_s}{L^2} r^3 + r^2 - R_s r = \frac{E^2}{L^2} r^4$$

New variable: $\quad x \equiv \dfrac{2L^2}{R_s r}$

$$\Rightarrow \left(\frac{dx}{d\phi}\right)^2 + \frac{4L^2}{R_s^2} - 2x + x^2 - \frac{R_s^2}{2L^2} x^3 = \frac{4E^2L^2}{R_s^2} \xRightarrow[\frac{d}{d\phi}]{} \boxed{\frac{d^2x}{d\phi^2} - 1 + x = \overbrace{\frac{3R_s^2}{4L^2} x^2}^{\text{absent in Newtonian grav.}}}$$

Perihelion advance

$$x \equiv \frac{2L^2}{R_s r}\,, \qquad \boxed{\frac{d^2x}{d\phi^2} - 1 + x = \underbrace{\alpha\, x^2}_{\text{absent in Newtonian grav.}}}\,, \qquad \alpha \equiv \frac{3R_s^2}{4L^2}$$



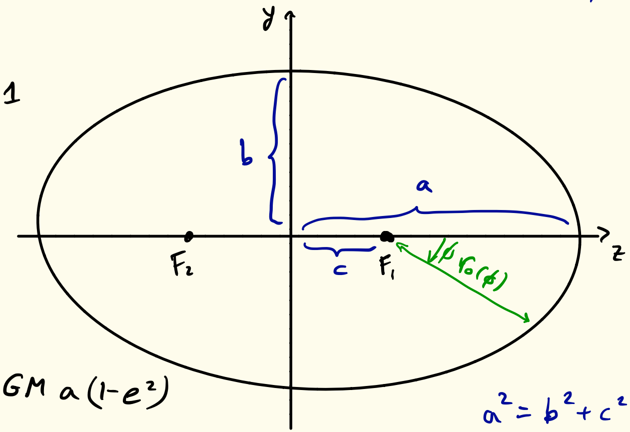
Let's solve this equation for $\alpha \ll 1$:

$$x(\phi) = x_0(\phi) + \alpha\, x_1(\phi) + \mathcal{O}(\alpha^2)$$

$$\Rightarrow \quad x_0(\phi) = 1 + \underset{\text{eccentricity}}{e} \cos\phi + \overset{\text{wlog}}{\cancel{d \sin\phi}} \quad \leftarrow \text{perfect ellipse}$$

$$r_0(\phi) = \frac{2L^2}{R_s}\,\frac{1}{1+e\cos\phi}$$

$$e^2 = 1 - \frac{b^2}{a^2} \qquad\qquad \left(\frac{z}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



$$a + c = r_0(\phi = \pi)$$

$$a - c = r_0(\phi = 0)$$

$$2a = \frac{2L^2}{R_s}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{4L^2}{R_s(1-e^2)} \Rightarrow L^2 = GM\,a(1-e^2)$$

$$a^2 = b^2 + c^2$$

Perihelion advance

$$x \equiv \frac{2L^2}{R_s r}\ , \qquad \boxed{\frac{d^2x}{d\phi^2} - 1 + x = \overbrace{\alpha\, x^2}^{\text{absent in Newtonian grav.}}}\ , \qquad \alpha \equiv \frac{3R_s^2}{4L^2}$$

Let's solve this equation for $\alpha \ll 1$:

$$x(\phi) = x_0(\phi) + \alpha\, x_1(\phi) + \mathcal{O}(\alpha^2)$$

$$\Rightarrow \quad x_0(\phi) = 1 + \underset{\uparrow}{e} \cos\phi$$

eccentricity

$$\Rightarrow \quad \frac{d^2x_1}{d\phi^2} + x_1 = x_0^2 = (1 + e\cos\phi)^2$$

[forced h.o.]

Exercise : check that

int. const.

$$x_1(\phi) = \underbrace{1 + \frac{e^2}{2}}_{\text{constant}} + \underline{e\,\phi \sin\phi} \underbrace{- \frac{1}{6} e^2 \cos 2\phi + C_1 \sin\phi + C_2 \cos\phi}_{\text{periodic function with period } 2\pi}$$

$$X = \underbrace{1 + e\cos\phi + \alpha e \phi \sin\phi}_{e\cos[(1-\alpha)\phi] + \mathcal{O}(\alpha^2)} + \alpha\, g(\phi) + \mathcal{O}(\alpha^2)$$

$g(\phi + 2\pi) = g(\phi)$

$\Rightarrow$ The perihelion advances by $\Delta\phi = 2\pi\alpha = \dfrac{6\pi\, G^2 M^2}{L^2}$

during each orbit

Using $L^2 \approx GM(1-e^2)a \rightarrow$

$$\boxed{\Delta\phi = \frac{6\pi\, GM}{(1-e^2)\, a}}$$

arc seconds

For Mercury, $\Delta\phi_{\text{Mercury}} \approx 5.01 \times 10^{-7}$ radians/orbit $\underset{\uparrow}{=}$ $43''$/century

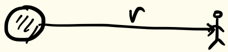
$T_{\text{orbit}} = 88$ days

[Notice that the effects of other planets lead to $\Delta\phi_{\text{Mercury}} \simeq 532''$/century

Observation: $\Delta\phi_{\text{Mercury}} = 575''$/century]

Gravitational Redshift

observer at fixed r [Not free falling]



photon $x^\nu(\lambda)$

$$U^\mu = (U^0, 0, 0, 0) \Rightarrow U^0 = \left(1 - \frac{R_s}{r}\right)^{-1/2}$$

$$U^\mu U^\nu g_{\mu\nu} = -1$$

observed frequency $= \omega = -g_{\mu\nu} U^\mu_{obs} \dfrac{dx^\mu}{d\lambda} =$

$$= \left(1 - \frac{R_s}{r}\right)^{1/2} \frac{dt}{d\lambda} = \left(1 - \frac{R_s}{r}\right)^{-1/2} E$$

conserved quantity

$$\Rightarrow \quad \frac{\omega_2}{\omega_1} = \left(\frac{1 - R_s/r_1}{1 - R_s/r_2}\right)^{1/2} \approx 1 - \frac{R_s}{2r_1} + \frac{R_s}{2r_2} = 1 + \Phi_1 - \Phi_2$$

Newtonian pot.

Schwarzschild Black Holes

$$ds^2 = \left(1 - \frac{R_s}{r}\right)\left[-dt^2 + \underbrace{\frac{dr^2}{\left(1-\frac{R_s}{r}\right)^2}}_{dr^{*2}}\right] + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

check!

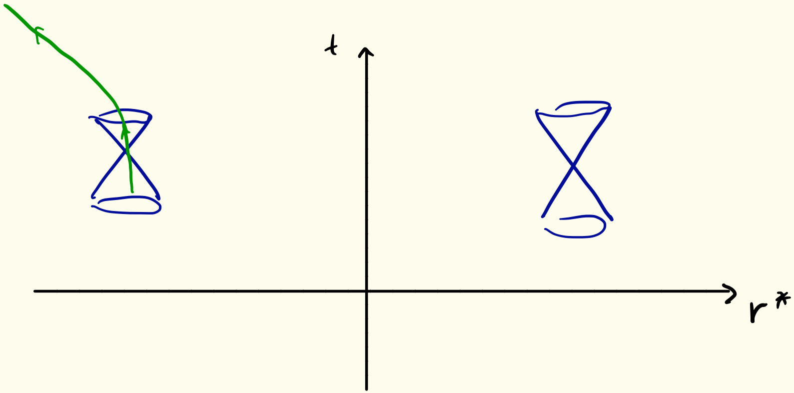
$\Rightarrow$ $$r^* = r + R_s \log\left(\frac{r}{R_s} - 1\right)$$ [Tortoise coordinate]

$$ds^2 = \left(1 - \frac{R_s}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2\, d\Omega^2$$

$r = R_s$

$\Updownarrow$

$r^* = -\infty$



Wisdom of the day

Deep Work

See calnewport.com
See also the book "Four Thousand Weeks"